Introduction to Cosmology

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Outline

Homogeneous and Isotropic Model

Homogenity: Means that universe looks the same at each point.

Isotropy: Means that universe looks the same in all directions.

These are two important properties of space which are independent of each other. But isotropy at each point implies homogenity also.

Cosmological principle: *Universe is homogeneous and isotropic at any given cosmic time.*

The cosmological principle is supported by the observational evidence that the universe becomes smooth at large scales. The cosmological principle presents the idealized picture of the universe. The departure from homogenity and isotropy is extremely important which led to the structure formation in the universe.

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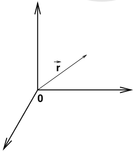
Hubble's law

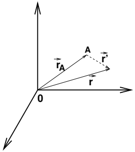
We examine the motion of matter in a coordinate system in which it is at rest at the origin. We now ask for the velocity dstribution consistent with homgenity and isotropy. Hubbles law:



$$\vec{v} = H(t)\vec{r} \tag{1}$$

Velocity field (1) is isotropic at O. Let us verify that (1) holds for any observer situated at a point A, The observer at A is in motion with respect to O.

$$\vec{r}' = \vec{r} - \vec{r}_A.$$



So,

$$\begin{aligned}
\vec{v}' &= \vec{v} - \vec{v}_A = H\vec{r} - H\vec{r}_A \\
\vec{v}' &= H(\vec{r} - \vec{r}_A) \\
\vec{v}' &= H\vec{r}'
\end{aligned}$$

Therefore, velocity distribution (1) is homogeneous and isotropic.

Local expansion

The distance between two arbitrary points changes as

$$\frac{d\vec{r}_{AB}(t)}{dt} = H(t)\vec{r}_{AB}$$

Therefore,

$$\vec{r}_{AB}(t) = \vec{r}_{AB}(t_0)\exp\left(\int_{t_0}^{t} H(t')dt'\right)$$

Remark: The dynamics will be decided by $H(t)$.
If $H(t) = const.$, then

$$\vec{r}_{AB}(t) = \vec{r}_{AB}(t_0)e^{(t-t_0)H}$$

$$\rho = \frac{M}{\frac{4\pi}{3}R^3}$$

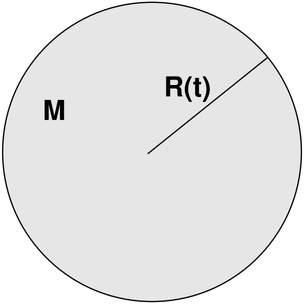
Differentiating w.r.t. time

$$\frac{d\rho(t)}{dt} = -\frac{3M}{[\frac{4\pi}{3}R^4]}\frac{dR}{dt}$$

But $dR/dt = v = HR$. Therefore,

$$\frac{d\rho(t)}{dt} = -\frac{3M}{[\frac{4\pi}{3}R^3]}H = -3\rho H$$

$$\frac{d\rho}{dt} = -3\rho H \tag{2}$$



The last can also be obtained from continuity equation

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot (\rho \vec{v})$$

ρ - function of time alone:

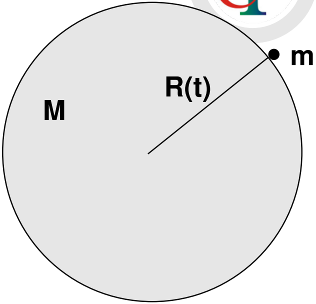
$$\frac{\partial \rho}{\partial t} = \rho H \vec{\nabla} \cdot \vec{r} = -3\rho H \tag{3}$$

$$\frac{\partial \rho}{\partial t} = \frac{d}{dt}\rho$$

Homogeneity and isotropy is a preserved property in time.

Outline

The acceleration of a particle with mass m due to the gravitational force of M is given by:



$$\frac{d^2R(t)}{dt^2} = \frac{-GM}{R^2(t)}$$

$$\frac{d^2R(t)}{dt^2} = \frac{d}{dt}(HR) = H\frac{dR}{dt} + R\frac{dH}{dt} = H^2R + R\frac{dH}{dt}$$

$$\frac{dH}{dt} = -\frac{GM}{R^3} - H^2$$

$$\frac{dH}{dt} = -H^2 - \frac{4\pi}{3}G\rho \tag{4}$$

Remark: If $H = const$, eqn (4) is inconsistent. Infact $H = const$ (with $\frac{dh}{dt} = \frac{\ddot{R}}{R} - H^2$) would imply $\ddot{R}(t) > 0$, which cannot come from eqn (4).

Multiply eqn (3) by $dR(t)/dt$:

$$\frac{dR}{dt}\left(\frac{d^2R}{dt^2}\right) = -\frac{GM}{R^2}\frac{dR}{dt}$$

$$\frac{1}{2}\frac{d}{dt}\left(\frac{dR}{dt}\right)^2 = \frac{d}{dt}\left(\frac{GM}{R}\right)$$

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{dR}{dt}\right)^2 - \frac{GM}{R}\right] = 0$$

$$\boxed{\frac{1}{2}\left(\frac{dR}{dt}\right)^2 - \frac{GM}{R} = A = const} \tag{5}$$

$$M = \frac{4}{3}\pi\rho R^3$$

Determining the constant A

$\{t_0, \rho_0\}$ present epoch and select a value of $R = R_0$ at $t = t_0$ for the sphere.

General character of the solution of (6)

At present, $\frac{dR}{dt} > 0$ implies that R was smaller in the past, but $\frac{8\pi G}{3}\rho R^2$ was larger, so $\frac{dR}{dt}$ was larger in the past:

$$\boxed{R(t_0) = 0, \qquad \left.\frac{dR}{dt}\right|_{t=t_0} = +\infty \qquad t = t_0}$$

Explanation

$$\boxed{R \geq 0 \text{ by definition}, \ddot{R}(t) < 0, \rho > 0}$$

Hence $R(t)$ was smaller and smaller as we go into past deeper and deeper, and dR/dt becomes larger and larger. Consequently, there was an epoch, say $t = 0$, when

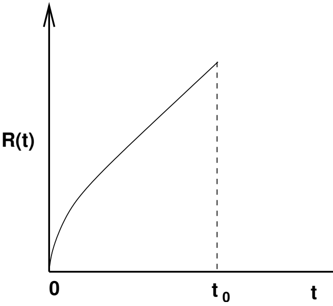
$$R(t = 0) = R(0) = 0 \qquad \left.\frac{dR}{dt}\right|_{t=0} = \infty$$

The prediction of the future depends upon the sign of $[\rho_0 - 3H_0/8\pi G]$ or how the present density compares with the critical density ρ_c:

$$\rho_c = \frac{3H_0^2}{8\pi G}$$

We also
defined the dimensionless density parameter Ω_0:

$$\Omega_0 \equiv \frac{\rho_0}{\rho} = \frac{8\pi G\rho_0}{3H_0^2}$$

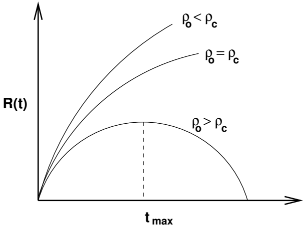


Classification of the solution

1. I $\rho_0 > \rho_c$ The second term in eqn (6) is positive. As R increases, the first term decreases and eventually becomes equal to the second term at a particular time. The RHS of eqn (6) then vanishes and expansion ceases, and contraction begins.
2. II $\rho_0 < \rho_c$ RHS of equation (6) is positive, leading to expansion forever.
   As $t \to \infty$, $R \to \infty$

$$\left.\frac{dR}{dt}\right|_{t=\infty} = \left[\frac{8\pi G}{3} R_0^2 (\rho_c - \rho)\right]^{1/2}$$

3. III $\rho_0 = \rho_c$ Expansion continues without bound.



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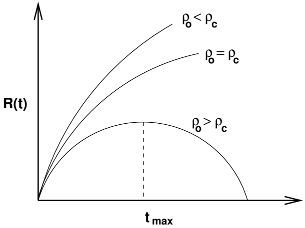

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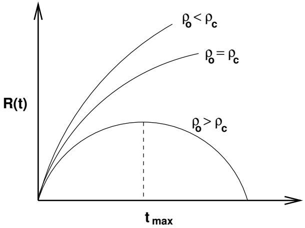

Maximum Age Estimate

$t = 0$, $R = 0$. Suppose the expansion rate is constant and given by the present value of Hubble parameter,

$$R(t_0) \equiv R_0 = \left(\frac{dR}{dt}\right)_{t=0} t_0 = H_0 R_0 t_0$$

$$\boxed{t_0 = \frac{1}{H_0}} \qquad \boxed{T_0 \simeq h^{-1} 9.8 \times 10^9 years}$$

$$H_0 \simeq 100 kms^{-1} mpc^{-1} \quad h \simeq 1 - 0.5 \quad t \simeq 9.8 \times 10^9 h^{-1} years$$

$$0.37 < H_0 t_0 < 1.47$$

Outline

Solutions of Evolution Equation

$\rho_0 = \rho_c$ or $\Omega_0 = 1$

$$\left(\frac{dR}{dt}\right)^2 = \frac{8\pi G}{3}\rho R^2 \qquad \boxed{\rho(t) = \frac{\rho_0}{R^3}R_0^3}$$

$$\left(\frac{dR}{dt}\right)^2 = \frac{8\pi G}{3}\frac{\rho_0 R_0^3}{R} \qquad dRR^{1/2} = const.dt$$

$$R(t) = R(t=0)(t/t_0)^{2/3} \qquad \Rightarrow \boxed{R_0\left(\frac{t}{t_0}\right)^{2/3} = R(t)}$$

$$R(t) \propto t^{2/3}$$

$$\dot{R}(t) \propto \frac{2}{3}t^{-1/3} \qquad \Rightarrow \qquad \frac{\dot{R}(t)}{R} = \frac{2}{3}\frac{1}{t}$$

Therefore, $t_0 = \frac{2}{3}\frac{1}{H_0}$.

$$\rho(t) = \frac{\rho_0 R_0^3}{R^3(t)} = \frac{\rho_0 R_0^3}{R_0^3(t/t_0)^2} = \frac{3H_0^2}{8\pi G t^2} t_0^2 = \frac{3H_0^2}{8\pi G t^2}\frac{4}{9}\frac{1}{H_0^2} = \frac{1}{6\pi G t^2}$$

$$\boxed{\rho(t) = \frac{1}{6\pi G t^2}}$$

Pressure corrections: Relativistic effects

$$\frac{\partial \rho}{\partial t} + 3H\left(\rho + \frac{P}{c^2}\right) = 0 \tag{7}$$

$$\frac{\ddot{R}}{R} = \frac{-4\pi G}{3}\left(\rho + \frac{P}{c^2}\right) \tag{8}$$

$$\frac{\dot{R}^2}{2} - \frac{4\pi G}{3} R^2 \rho = A \tag{9}$$

Consistency check: Differentiating (9), we get

$$\dot{R}\ddot{R} - \frac{4\pi G}{3} R^2 \dot{\rho} - \frac{4\pi G}{3} \times 2 \times \rho \dot{R} R = 0$$

$\dot{R} = HR$ lead to

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System (7), (8) and (9) are consistent.

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Equation of State: Out of the eqs (7), (8) and (9), only two are independtent. Eq. (9) can be obtained from (7) and (8). These eqs can be solved and $\rho(t)$, $R(t)$ can be uniquely determined provided the *equation of state* (= relation between ρ and P) is given. This relation, in simple cases, can be written as

$$P = \omega \rho c^2, \qquad \omega = \begin{cases} 0 & Dust \\ \frac{1}{3} & Radiation \\ -1 & Cosmological\ constant \end{cases}$$

$\omega = \frac{1}{3}$, $\rho_0 = \rho_c$

$$\frac{\partial \rho}{\partial t} + 3H\left(\rho + \frac{P}{c^2}\right) = 0 \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} + 4H\rho = 0$$

$$\boxed{\rho(t) = \rho_0 \frac{R_0^4}{R^4(t)}}$$

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho \qquad \Rightarrow \qquad \dot{R}(t) \propto \frac{1}{R(t)}$$

$$\boxed{R(t) = R_0 \left(\frac{t}{t_0}\right)^{1/2}}$$

$$H(t) = \frac{1}{2t} \qquad \Rightarrow \qquad \boxed{t_0 = \frac{1}{2H_0}}$$

$$\rho(t) = \frac{3}{32\pi G}\frac{1}{t^2}$$

Outline

Comoving Coordinates:

These coordinates are are carried along with the expansion. Since the expansion is uniform, the relation between the Physical and Comoving Coordinates is given by

$$\vec{r}(t) = a(t)\vec{x} \qquad\qquad a(t) \rightarrow \textit{scale factor}$$

The uniformity of expansion is encoded in the scale factor.

$$\left(\frac{dR}{dt}\right)^2 = \frac{8\pi G}{3}\rho R^2 - \frac{8\pi G}{3}R_0^2[\rho_0 - \rho_c]$$

Put $R(t) = a(t)x$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{H}{x^2a^2}$$

Let $Kc^2 = \frac{A}{x^2}$ or $K = \frac{A}{x^2 c^2}$. Dimensionally

$$[K] = \frac{L^2}{T^2} \frac{1}{L^2 \frac{L^2}{T^2}} = L^{-2}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2}$$

Also, from (8)

$$\frac{\ddot{a}}{a} = \frac{-4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)$$

Putting $c = 1$

$$\boxed{\frac{\partial \rho}{\partial t} + 3H(\rho + P) = 0}$$

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}}$$

$$\boxed{\frac{\ddot{a}}{a} = \frac{-4\pi G}{3}(\rho + 3P)}$$

For $K = 0$, $\rho_r(t) = \frac{3}{32\pi G t^2}$

$$\rho_r(t) = \alpha T^4$$

$$\boxed{T = \left(\frac{3}{32\pi G\alpha}\right)^{1/4} t^{-1/2}}$$

$$\boxed{T_K = 1.5 \times 10^{10} t_{sec}^{-1/2}}$$

$$\rho_r(t) \propto \frac{1}{a^4} \qquad \Rightarrow \qquad T(a) = T_0 \left(\frac{a_0}{a}\right)$$

Decoupling

Decoupling takes place at a temperature equal to the binding energy of Hydrogen atom.

$$3k_BT_d = 13.6eV$$

$$T_d = \frac{13.6eV}{3k_B} \approx 5 \times 10^4 K \quad (1eV \simeq 10^4 K)$$

However, temperature in reality is much smaller than this

$$T_d \simeq 3000K \qquad \Rightarrow \qquad \frac{a_0}{a_d} = \frac{3000}{T_0} \simeq 1000$$

Since $T_K = 1.5 \times 10^{10} T^{-1/2}$, decoupling time

$$t_d = 10^{13}\ sec \simeq 3 \times 10^5\ yrs.$$

Remark:

$$
\begin{aligned}
\frac{\Omega_r}{\Omega_m} &\propto \frac{1}{a} \\
&= \frac{\Omega_r^0}{\Omega_m^0}\frac{1}{a} \approx \frac{4\times 10^{-5}}{\Omega_m^0}\frac{1}{a}
\end{aligned}
$$

$$
\begin{aligned}
a_{eq} &\simeq 2.4\times 10^4 \Omega_0 \\
t_{eq} &\simeq 2.5\times 10^3 \Omega_0^{-3/2}\ years
\end{aligned}
$$

Cosmological Constant

Einstein introduced the *cosmological constant* to make the universe static (later described by him as his "biggest blunder").

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} + \frac{\Lambda}{3}$$

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} \qquad P_\Lambda = \frac{-\Lambda}{8\pi G}$$

$\Lambda > 0$ and ρ: $H = 0$

$$a(t) \propto e^{\frac{\Lambda}{3}t}$$

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3}\left(\rho_\Lambda + 3P_\Lambda\right) = +\frac{8\pi G}{3}\rho_\Lambda$$

$\ddot{a} > 0 \quad \Rightarrow$ accelerated expansion. **Inflation**

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} + \frac{\Lambda}{3}$$

$$1 = \Omega_m + \Omega_\Lambda - \frac{K}{a^2 H^2}$$

$$\boxed{\Omega_m + \Omega_\Lambda - 1 = \frac{K}{a^2 H^2}}$$

For Further Reading

J. V. Narlikar
An Introduction to Cosmology
(Cambridge University Press, 2002)

Steven Weinberg
Cosmology
(Oxford University Press)