AN UNDERGRADUATE'S GUIDE TO QUANTUM ENTANGLEMENT

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On an undergraduate level, one is made familiar to the wave function that contains the necessary information one requires to describe a particle in the microscopic world. When one talks about classical mechanics, the state of a particle can be described by specifying the position and momentum of the particle at that instant of time. However as the picture gets reduced to a smaller and smaller scale, it has been found that determination of position and momentum simultaneously is not possible – this fact has been inscribed in the Heisenberg's Uncertainty Principle. Thus we employ what is called the wave function of a particle for the business.

Let us denote the wave function of a particle by $\Psi(x)$. This is a complex quantity, but it has no physical significance of its own. It needs to be operated on by operators or integrated over some interval in space after multiplying with its complex conjugate (probability of finding the particle in that interval) to yield some meaning. However, it represents an independent state of a particular particle.

PRE-REQUISITES

SUPERPOSITION

Let Ψ be the wave function of a particle. It can be expressed as a linear combination of a number of 'eigenstates' of an observable in which the particle can exist, i.e.,

$$\Psi(x)= a_1\Psi_1(x) + a_2\Psi_2(x) + a_3\Psi_3(x) + \ldots + a_n\Psi_n(x)$$

Where, Ψ_1, Ψ_2, ..., Ψ_n are the eigenstates (possible states in which the observable has a well defined value), and a_1, a_2, ..., a_n are certain constants.

MEASUREMENT

- When a measurement is done on the particle, the wave function *collapses* to a single eigenstate (say Ψ_2), which physically means we have found the particle to be in state Ψ_2 after measurement, and thus after measurement $\Psi(x) = \Psi_2(x)$. This happens with a probability $|a_2|^2$.
- When we operate an operator A on Ψ, we obtain a value called the eigenvalue.

A$\Psi = \lambda\Psi$

Where, λ is the eigenvalue, and is the value we would obtain if we were to actually measure A. However, operating A on Ψ does not imply a measurement of A on Ψ by itself.

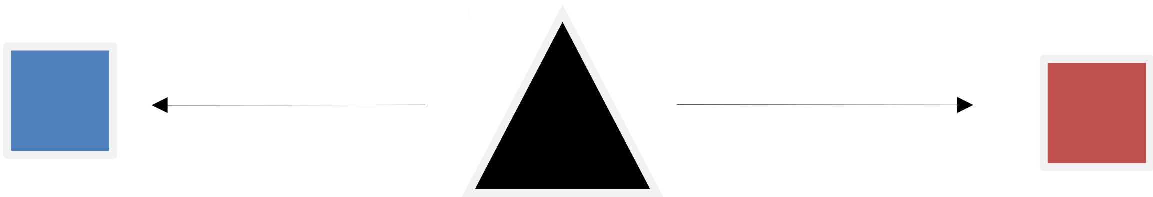
INTRODUCTION TO ENTANGLEMENT

Now let us consider a wave function $[\Psi_a(x_1)\phi_a(x_2) + \Psi_b(x_1)\phi_b(x_2)]$, where Ψ_a , Ψ_b are possible states of particle 1, and ϕ_a , ϕ_b are the possible states of 2. This wave function gives the probability density of both particles simultaneously in space. However, one can notice that the individual wave functions of the particles Ψ's, ϕ's cannot be factored out. These are thus called an entangled pair of particles. What we mean by entanglement is that one particle is influenced by the other, irrespective of the distance by which the particles are separated.

Suppose we make a measurement A on particle 1, which can distinguish between Ψ_a and Ψ_b, while particle 2 is elsewhere in space. Now, the measurement will result in the collapse of the wave function which will yield an eigenvalue corresponding to one of the states (either Ψ_a or Ψ_b) in which particle 1 can exist. Suppose we realise the eigenvalue obtained corresponds to state Ψ_a of particle 1. We can immediately predict that the state of particle 2 is ϕ_a and thus we can also predict what eigenvalue the same measurement would yield if performed on particle 2.

Suppose, in our measurement A, instead of Ψ_a we obtain the eigenvalue that corresponds to the state Ψ_b. At that very instant we can, with assurance, say that particle 2 is in state ϕ_b, in accordance to the wave function for entangled particles. Thus the state of particle 2 changes when measurement is made on particle 1. Now if we assume that both the particles are separated by a distance greater than what light can travel in 1 sec (greater than c), this influence will still be observed. This seems to indicate the occurrence of a faster than light influence, which seems to defy relativity. This obviously made Einstein uncomfortable.

EPR (EINSTEIN-PODOLSKY-ROSEN) PARADOX – THE SIMPLEST NOTION



D1 **P** **D2**

Let us consider this extremely simplified experiment. P is a generator of entangled particles (say, there may occur some reaction which produces electrons and positrons from unstable nuclei, which are entangled pairs of particles). D1 and D2 are two detectors which detect the oncoming particles and measure their spins. Let us assume that one pair of entangled particles are generated by P at a time which in turn are detected by D1 and D2. Let the particles detected by D1 and D2 be denoted by 1 and 2 respectively. Also, D1 and D2 are placed miles apart.

We start the experiment. Entangled pair of particles 1 and 2 is generated by P. We first measure the z component of spin of both the particles.

If Iz;+> denotes z component of spin +1/2 and Iz;-> denotes z component of spin -1/2,

Possible states of particle 1 are $Iz;+>_1$, $Iz;->_1$

Possible states of particle 2 are $Iz;+>_2$, $Iz;->_2$

We already know that the wave function governing the two particles is

$\Psi= (Iz;+>_1\ Iz;->_2 + Iz;->_1\ Iz;+>_2)/\surd 2$.

D1 detects $Iz;+>_1$ for particle 1. We immediately know that D2 would detect $Iz;->_2$ for particle 2 without looking at the result of measurement by D2. This seems like the discovery of spin of particle 1 has locked and signalled particle 2 to assume the spin state $Iz;->_2$.

Now let Ix;+> denote x component of spin +1/2 and Ix;-> denote x component of spin -1/2.

The z and x components of spin are known to be related as

$Ix;+> = (Iz;+> + Iz;->)/\surd 2$

$Ix;-> = (Iz;+> - Iz;->)/\surd 2$

Thus by some simple mathematical manipulation, we can write Ψ as

$\Psi= 1/\surd 2(Ix;+>_1\ Ix;->_2 + Ix;->_1\ Ix;+>_2)$

Now the detectors are made to detect the x component of spin.

D1 detects $Ix;->_1$ for particle 1. We immediately realise that the x component of spin of particle 2 is $Ix;+>_2$. This also seems like the discovery of spin of particle 1 has generated a signal to communicate to particle 2 to assume the spin state $Ix;+>_2$.

However for this to happen, there needs to exist a faster than light influence that is governing the entire phenomenon, as the detectors are placed miles apart. This is not in accordance with relativity.

Thus, Einstein suggested something very seemingly obvious that the z and x components of the particles that we found on measurement were preordained, they already existed. It was just our ‘ignorance’ that we did not measure them earlier; they were waiting to be discovered. That, had we measured them at a time other than now, we would have obtained the same values and there is nothing strange about it - this is how the macroscopic world works.

All the z, x, y components of spin angular momentum of a particle have definite predetermined values is what Einstein tried to suggest.

However, one of the most fundamental pillars of Quantum Mechanics, the Heisenberg Uncertainty Principle, says that the position and momentum of a microscopic particle cannot be determined simultaneously.The same holds true for the x and z components of a spin – they cannot be determined simultaneously. But here we can know the values of either of the two spin components of particle 2, without ever disturbing it, simply by measuring corresponding spin-components of particle 1.

This is a contradiction.

This led Einstein to consider that maybe Quantum Mechanics is incomplete, and this “spooky action at a distance” involves some “hidden variables” that need be to be discovered in order to obtain the full picture.

BELL’S THEOREM

In 1964, John Bell contested the idea of hidden variables with a seemingly simple logic. Let us try to understand it in a simpler manner as follows.

Let us accept the fact that hidden variables exist, for now. We give Alice and Bob one particle each and ask them to measure properties P, Q and R. Let the measured values of P, Q and R be p, q, r respectively. Also, let us make another assumption that p, q, r can take values + or – only.

Now, p, q and r already have pre-existing values, they only ‘remain to be discovered’, i.e., their values have already been decided by nature and they exist even when we are not making the measurement (hidden variables). More technically, we can say that values of P, Q and R are local.

Since there are only two possible values the hidden values can take, total number of possible combinations is $2^3 = 8$.

Alice and Bob start making measurements and they measure any property P, Q or R randomly at will. Irrespective of which property they measure, if the value of the hidden variable obtained match for both we call it 1, else if they do not match we call it 0.

If suppose the hidden variables contain the values

p= +

q= +

r= +

and if the situation where Alice measures P and Bob measures Q is represented as PQ, then [PQ]=1, since p=q= +. Since we are dealing with entangled particles, we can safely assume [PP]=[QQ]=[RR]=1, since the corresponding values of the properties would be same. Hence, we take only those cases where Alice and Bob measure different properties at one time.

We can expect to obtain the following results.

p	**q**	**r**	**[PQ]**	**[QR]**	**[PR]**	**([PQ]+[QR]+[PR])/3 (Probability of occurrence)**
+	+	+	1	1	1	1
+	+	-	1	0	0	1/3
+	-	-	0	1	0	1/3
-	-	-	1	1	1	1
+	-	+	0	0	1	1/3
-	+	+	0	1	0	1/3
-	+	-	0	0	1	1/3
-	-	+	1	0	0	1/3

Thus, probability of getting matching values of measured values ≥ 1/3

([PQ]+[QR]+[PR])/3 ≥ 1/3

$\boxed{[PQ]+[QR]+[PR] \geq 1}$ **This is the famous <u>BELL'S INEQUALITY</u>.**

This is profound. Bell, with some fairly simple experimental logic, said that for hidden variables to exist, i.e., for properties to have pre-determined values, the above inequality must be satisfied.

<u>QUANTUM TREATMENT TO BELL'S INEQUALITY</u>

Let us try to write the wave functions for the two particles.

$|p;+\rangle$ denotes p=+

$|p;-\rangle$ denotes p= -

$|q;+\rangle$ denotes q=+

$|q;-\rangle$ denotes q= -

$|r;+\rangle$ denotes r=+

$|r;-\rangle$ denotes r= -

The two particles are entangled, and the properties P, Q, R are certain components of the spin, respectively. Drawing analogy from the wave function mentioned above for entangled particles, we have –

$\Psi= 1/\sqrt{2}(|p;+\rangle_1 |p;+\rangle_2 + |p;-\rangle_1 |p;-\rangle_2)= 1/\sqrt{2}(|q;+\rangle_1 |q;+\rangle_2 + |q;-\rangle_1|q;-\rangle_2)$

$= 1/\sqrt{2}(|r;+\rangle_1 |r;+\rangle_2 + |r;-\rangle_1 |r;-\rangle_2)$

Here, the subscripts 1 and 2 refer to particles 1 and 2 respectively.

We can choose here that the P, Q and R are defined by the following –

$|q;+\rangle_1 = 1/2|p;+\rangle_1 + \sqrt{3}/2|p;-\rangle_1$

$|q;-\rangle_1 = \sqrt{3}/2|p;+\rangle_1 - 1/2|p;-\rangle_1$

$|r;+\rangle_1 = 1/2|p;+\rangle_1 - \sqrt{3}/2|p;-\rangle_1$

$|r;-\rangle_1 = \sqrt{3}/2|p;+\rangle_1 + 1/2|p;-\rangle_1$

Now the probability of getting matching result for P measurement on particle 1 and Q measurement on particle 2, is given by

$[PQ] = | <p;+|_1 <q;+|_2 |\Psi> |^2 + | <p;-|_1 <q;-|_2 |\Psi> |^2$

which is just the sum of probabilities of getting + for both and – for both. A typical term may be calculated as follows

$<p;+|_1 <q;+|_2 |\Psi> = <p;+|_1 <q;+|_2 (|p;+>_1 |p;+>_2 + |p;->_1 |p;->_2)/\sqrt{2}$

$= ({}_1<p;+|p;+>_1 {}_2<q;+|p;+>_2 + {}_1<p;+|p;->_1 {}_2<q;+|p;->_2)/\sqrt{2}$

$= {}_2<q;+|p;+>_2/\sqrt{2}$

Remembering that $|q; +>_2 = (1/2)|p;+>_2 + (\sqrt{3}/2)|p;->_2$, we can calculate the above as

${}_1<p;+| {}_2<q;+| |\Psi> = ((1/2){}_2<p;+| + (\sqrt{3}/2){}_2<p;-|)|p;+>_2/\sqrt{2} = 1/(2\sqrt{2})$.

Performing the above experiment now, we get

$[PQ] = | {}_1<p;+| {}_2<q;+| |\Psi> |^2 + | {}_1<p;-| {}_2<q;-| |\Psi> |^2 = 1/4$

$[QR] = | {}_1<q;+| {}_2<r;+| |\Psi> |^2 + | {}_1<q;-| {}_2<r;-| |\Psi> |^2 = 1/4$

$[PR] = | {}_1<r;+| {}_2<p;+| |\Psi> |^2 + | {}_1<p;-| {}_2<q;-| |\Psi> |^2 = 1/4$

$[PQ] + [QR] + [PR] = 3/4 < 1$

Thus, Bell's inequality is violated.

This is a situation which can be possible in the real world since no basic laws of nature are violated, and it does not satisfy Bell's inequality.

Therefore, the possibility of the existence of hidden variables is ruled out. What it means is that **quantum mechanics is non-local**. This means quantum mechanical properties are not pre determined, they randomly arrive at any one of the possible eigenvalues on measurement, and there is no rule to predict what value they would take. **There are no hidden variables.**

Thus Bell postulated:

No physical theory of local hidden variables can ever reproduce all the predictions of quantum mechanics.

This is true. This has been experimentally verified later, with the first actual Bell test conducted by Freedman and Clauser in 1972, and many more experiments later.

However, this fact does not establish that there is some faster than light influence that exists to communicate between two entangled particles. Alice cannot select a state for her particle at will, nor can the state of her particle force the Bob's particle to attain a state of Alice's choice. This just happens randomly, once Alice knows the state of her particle she can predict what state Bob's particle would be in from the wavefunction that governs the two entangled states, but nothing can be induced by either Alice or Bob to obtain a measurement of their choice.

This behaviour at a distance indeed seems 'spooky' but it seems nature has its own ways.

CONCLUSION

An extremely simplified and understandable introduction to Quantum Entanglement, by only accentuating the amount of knowledge one is expected to possess in an undergraduate level, is aimed at in this project. It is one of the most bizarre and baffling phenomena to have been observed, and we do not yet have an explanation as to why such a thing should occur in the microscopic world, or why this phenomena would not manifest macroscopically. All we know is that entangled particles truly exist, and every time nature only shouts out to us that however unapparent quantum results be, quantum theory is accurate and nature chooses to behave this way.

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